EXAMPLE 2.11 A thin rectangular homogeneous thermally conducting plate lies in the xy-plane defined by $0 \le x \le a$, $0 \le y \le b$. The edge $y = 0$ is held at the temperature $Tx(x-a)$, where T is a constant, while the remaining edges are held at 0°. The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate.

Solution Since no heat sources and sinks are present in the plate, the steady state temperature u must satisfy $\nabla^2 u = 0$. Hence the problem is to solve

$$\text{PDE: } \nabla^2 u = 0$$

$$\text{BCs: } u(0, y) = 0, \qquad u(a, y) = 0, \qquad u(x, b) = 0, \qquad u(x, 0) = Tx(x-a)$$

This is a typical Dirichlet's problem. The general solution satisfying the first three BCs is given by Eq. (2.47). Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

where

$$A_n \sinh\frac{-n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx$$

Using the last BC: $u(x, 0) = Tx(x-a) = f(x)$, we get

$$A_n \sinh\frac{-n\pi b}{a} = \frac{2}{a}\int_0^a Tx(x-a) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{2T}{a}\int_0^a x(x-a) \sin\left(\frac{n\pi}{a}x\right) dx$$

$$= -\frac{a}{n\pi}\cdot\frac{2T}{a}\left[\int_0^a x(x-a)\, d\left\{\cos\left(\frac{n\pi}{a}x\right)\right\}\right]$$

$$= -\frac{2T}{n\pi}\left[(x-a)\cos\left(\frac{n\pi}{a}x\right)\right]_0^a - \frac{a}{n\pi}\int_0^a (2x-a)\, d\left[\sin\left(\frac{n\pi}{a}x\right)\right]$$

$$= \frac{2aT}{n^2\pi^2}\left[(2x-a)\sin\left(\frac{n\pi}{a}x\right)\right]_0^a - \int_0^a 2\sin\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{2aT}{n^2\pi^2}\left\{a \sin n\pi + \frac{2a}{n\pi}\left[\cos\left(\frac{n\pi}{a}x\right)\right]_0^a\right\}$$

$$= \frac{2aT}{n^2\pi^2}\,\frac{2a}{n\pi}(\cos n\pi - 1) = \frac{4a^2T}{n^3\pi^3}[(-1)^n - 1]$$

Thus the required temperature distribution is given by

$$u(x, y) = \sum_{n=1}^{\infty} \operatorname{cosech}\left(-\frac{n\pi}{a}b\right)\frac{4Ta^2}{n^3\pi^3}[(-1)^n - 1]\sin\left(\frac{n\pi}{a}x\right)\sinh\left[\frac{n\pi}{a}(y-b)\right]$$

EXAMPLE 2.12 Solve

$$\nabla^2 u = 0, \qquad 0 \le x \le a, \qquad 0 \le y \le b$$

satisfying the BCs:

$$u(0, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, b) = 0$$

$$\frac{\partial u}{\partial x}(a, y) = T\sin^3\frac{\pi y}{a}$$

Solution Using the variables separable method, one of the acceptable general solutions is given by Eq. (2.38). Hence

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the BC: $u(x, 0) = 0$, we get

$$0 = c_3(c_1 e^{px} + c_2 e^{-px})$$

implying $c_3 = 0$. Therefore,

$$u(x, y) = c_4 \sin py (c_1 e^{px} + c_2 e^{-px})$$

Now, using the BC: $u(x, b) = 0$, we obtain

$$0 = c_4 \sin pb (c_1 e^{px} + c_2 e^{-px})$$

$c_4 \ne 0$ (why?) implying $\sin pb = 0$ which gives

$$pb = n\pi \quad \text{or} \quad p = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

Thus,

$$u(x, y) = c_4 \sin\left(\frac{n\pi}{b}y\right)(c_1 e^{px} + c_2 e^{-px})$$

Renaming the constants, we have

$$u(x, y) = \sin\left(\frac{n\pi}{b}y\right)\left[A\exp\left(\frac{n\pi}{b}x\right) + B\exp\left(-\frac{n\pi}{b}x\right)\right], \qquad n = 1, 2, \ldots$$

If we use the BC: $u(0, y) = 0$, we get

$$0 = \sin\left(\frac{n\pi}{b}y\right)(A + B)$$

giving $A + B = 0$; therefore, $A = -B$. Thus,

$$u(x, y) = A \sin\left(\frac{n\pi}{b} y\right)\left[\exp\left(\frac{n\pi}{b} x\right) - \exp\left(-\frac{n\pi}{b} x\right)\right]$$

$$= 2A \sin\left(\frac{n\pi}{b} y\right) \sinh\left(\frac{n\pi}{b} x\right), \qquad n = 1, 2, \ldots$$

Differentiating with respect to x, we obtain

$$\frac{\partial u}{\partial x} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} x\right)$$

The last BC yields

$$T \sin^3 \frac{\pi y}{a} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} a\right)$$

from which we can determine $2A$. Hence, the required solution is

$$u(x, y) = \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech} \frac{n\pi}{b} a \sinh\left(\frac{n\pi}{b} x\right)$$

The principle of superposition gives the required solution as

$$u(x, y) = \sum_{n=1}^{\infty} \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech}\left(\frac{n\pi}{b} a\right) \sinh\left(\frac{n\pi}{b} x\right)$$

EXAMPLE 2.16 A thin annulus occupies the region $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at $T = K \cos(\theta/2)$, where K is a constant. Determine the temperature distribution in the annulus.

Solution Mathematically, the problem is to solve

$$\text{PDE:}\quad \nabla^2 T = 0, \qquad a \le r \le b,\ 0 \le \theta \le 2\pi$$

$$\text{BCs:}\ T(a,\theta) = 0$$

$$T(b,\theta) = k \cos \theta/2$$

The required general solution is given by Eq. (2.57) in the form

$$T(r,\theta) = (c_1 r^n + c_2 r^{-n})(c_3 \cos n\theta + c_4 \sin n\theta)$$

Using the first BC, we get

$$0 = (c_1 a^n + c_2 a^{-n})(c_3 \cos n\theta + c_4 \sin n\theta)$$

implying thereby $c_1 a^n + c_2 a^{-n} = 0$, or $c_2 = -c_1 a^{2n}$. After adjusting the constants suitably, we have

$$T(r,\theta) = \left(r^n - \frac{a^{2n}}{r^n}\right)(A \cos n\theta + B \sin n\theta)$$

The principle of superposition gives

$$T(r,\theta) = \sum_{n=1}^{\infty} \left(r^n - \frac{a^{2n}}{r^n}\right)(A_n \cos n\theta + B_n \sin n\theta)$$

Now, using the second boundary condition, we obtain

$$T(b,\theta) = K \cos\frac{\theta}{2} = \sum_{n=1}^{\infty} (b^n - b^{-n}a^{2n})(A_n \cos n\theta + B_n \sin n\theta)$$

which is a full-range Fourier series. Hence,

$$A_n(b^n - b^{-n}a^{2n}) = \frac{1}{\pi}\int_0^{2\pi} K \cos\frac{\theta}{2} \cos n\theta \, d\theta$$

$$= \frac{k}{2\pi}\int_0^{2\pi}\left[\cos\left(n+\frac{1}{2}\right)\theta + \cos\left(n-\frac{1}{2}\right)\theta\right] d\theta$$

$$= \frac{k}{2\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\sin\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= 0$$

implying $A_n = 0$. Also,

$$B_n(b^n - b^{-n}a^{2n}) = \frac{k}{\pi}\int_0^{2\pi} \cos\frac{\theta}{2} \sin n\theta \, d\theta$$

$$= \frac{k}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)\theta + \sin\left(n-\frac{1}{2}\right)\theta\right] d\theta$$

$$= -\frac{k}{2\pi}\left[\frac{\cos\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= -\frac{k}{2\pi}\left(-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right)$$

$$= \frac{k}{\pi}\left(\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right) = \frac{k}{\pi}\frac{2n}{n^2 - \frac{1}{4}}$$

or

$$B_n(b^n - b^{-n}a^{2n}) = \frac{8kn}{\pi(4n^2-1)}$$

Thus the temperature distribution in the annulus is given by

$$T(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\left[\frac{(r/a)^n - (a/r)^n}{(b/a)^n - (a/b)^{-n}}\right]\sin n\theta$$